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On the adiabatic theorem for non-self-adjoint Hamiltonians

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Abstract. We consider the evolution of a two-level system driven by a non-self-adjoint Hamiltonian $H(\epsilon t)$ and treat the adiabatic limit $\epsilon \to 0$. While adiabatic theorem-like results do not hold true in general for this case, we prove that they are still valid for the subspace corresponding to the eigenvalue having the largest imaginary part (least dissipative eigenvalue). The theory gives the full asymptotic expansion of the evolution restricted to this subspace. The first correction beyond Berry's phase is to our best knowledge given explicitly for the first time.

1. Introduction

The adiabatic approximation for dissipative systems appears naturally in atomic physics and quantum optics. The interest in the special case dealing with only a finite number (e.g. two) of states has been renewed by [1–3] (see also [4]). Most of the papers are concerned with the generalization of Berry's phase to non-self-adjoint Hamiltonians and the validity of the adiabatic theorem is taken for granted. There are two regimes to be discussed. The first one is the 'weak non-Hermiticity' regime in which the absolute values of the imaginary parts of the eigenvalues are of the same order of magnitude as the slowness parameter (which we call ϵ in what follows). In this case (as has been proven in [3]) a complete generalization of the adiabatic theorem for non-degenerate eigenvalues is possible; Berry's phase (complex in general) and the transition probabilities can be computed. If degeneracies occur, the situation is more subtle (due to the conflicting demands between the adiabatic approximation and the experimental requirement that the signal should not be completely cancelled by the dissipation); we refer the reader to [2] for the discussion of this interesting case which seems to deserve further study.

The second regime is the 'strong non-Hermiticity' one in which at least some of the eigenvalues have imaginary parts much larger (in absolute value) than the slowness parameter. In this case (see the discussion below) a complete generalization of the adiabatic theorem seems not to be possible. However, in this paper we prove an adiabatic theorem-like result for the strong non-Hermiticity regime. More exactly we will prove that an adiabatic expansion exists for the evolution restricted to the subspace corresponding to the least dissipative eigenvalue (i.e. the one having the largest imaginary part), which is assumed to be isolated.

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For simplicity we only treat a two-level system. For n-level systems ($n \geqslant 3$) with non-degenerate spectrum the proof carries over directly. With some additional technicalities the result can also be extended to more general situations. The theory we have developed permits us to calculate the whole asymptotic expansion in ϵ, having as the leading term the result given in [1]; in addition we give the explicit formula for the first-order correction.

We make two further remarks. Firstly, we do not require a definite sign for the imaginary parts of the eigenvalues; thus our results also apply to systems with pumping. Of course the considered eigenvalue should not be strongly dissipative in order to avoid the cancellation of the signal. Secondly, since in the non-self-adjoint case the unitarity of the evolution is lost, our results do not imply results about the transition probabilities.

As already remarked above, in the strong non-Hermiticity regime, if the considered eigenvalue is not the least dissipative one, it seems that the adiabatic approximation as used in [1] does not hold.

To see this we consider a non-self-adjoint time-dependent Hamiltonian $H(\epsilon t) \neq H^\dagger(\epsilon t)$ (with $\epsilon > 0$) in a two-dimensional Hilbert space. We introduce the variable $s \equiv \epsilon t$. In the following we will be interested in the limit $\epsilon \to 0$ whilst s is kept fixed and finite. We write $H(s)$ in the form given in [1]:

$$H(s) = \sum_{j=1}^{2} E^{(j)}(s)|\hat{\psi}^{(j)}(s)\rangle\langle\hat{\theta}^{(j)}(s)| \tag{1.1}$$

where it is assumed that $\langle\hat{\theta}^{(j)}(s)|\hat{\psi}^{(k)}(s)\rangle = \delta_{jk}$ so that $E^{(j)}(s)$ are the eigenvalues of $H(s)$.

Let $|\Psi(s)\rangle$ be a solution of the time-dependent Schrödinger equation (we often abbreviate $\partial_s \equiv \dot{\ }$)

$$\mathrm{i}\epsilon|\dot{\Psi}(s)\rangle = H(s)|\Psi(s)\rangle\,. \tag{1.2}$$

Expanding

$$|\Psi(s)\rangle = \sum_{j=1}^{2} c^{(j)}(s)\exp\left(-\frac{\mathrm{i}}{\epsilon}W^{(j)}(s)\right)|\hat{\psi}^{(j)}(s)\rangle \tag{1.3}$$

we get (see e.g. [1])

$$\begin{aligned}\dot{c}^{(l)}(s) &+ \langle\hat{\theta}^{(l)}(s)|\partial_s\hat{\psi}^{(l)}(s)\rangle c^{(l)}(s) \\ &= -\sum_{j\neq l}\langle\hat{\theta}^{(l)}(s)|\partial_s\hat{\psi}^{(j)}(s)\rangle c^{(j)}(s)\exp\left(-\frac{\mathrm{i}}{\epsilon}[W^{(j)}(s) - W^{(l)}(s)]\right).\end{aligned} \tag{1.4}$$

Here we have used the abbreviation

$$W^{(i)}(s) = \int_{s_0}^{s} E^{(i)}(u)\,\mathrm{d}u\,.$$

Equation (1.4) is usually taken as the starting point to justify the lowest-order adiabatic approximation (see e.g. [1]): For $H(s) = H^\dagger(s)$ the $E^{(j)}(s)$ are real and the right-hand side of (1.4) can be neglected in the limit $\epsilon \to 0$ because of the resulting rapid oscillations. For $H(s) \neq H^\dagger(s)$ the situation is more involved. Assume for definiteness that

$$\mathrm{Im}[E^{(2)}(s) - E^{(1)}(s)] < 0 .$$

The right-hand side of (1.4) can then only safely be neglected for the equation corresponding to $l = 1$. For $l = 2$ the right-hand side blows up as $\exp(\mathrm{const}\,\epsilon^{-1})$ in the limit $\epsilon \to 0$. Since the system of differential equations is coupled the justification of the adiabatic approximation for both $c^{(l)}$ needs a more careful examination.

From the results in section 2 it follows in particular that if $H(s)$ is constant outside the interval $[s_-, s_+]$, $s_0 \leqslant s_-$, $s \geqslant s_+$ and

$$\mathrm{Im}[E^{(2)}(u) - E^{(1)}(u)] \leqslant 0 \qquad s_0 \leqslant u \leqslant s$$

then

$$\begin{aligned} c^{(1)}(s) = c^{(1)}(s_0) \exp \int_{s_0}^{s} \mathrm{d}u \{ &-\langle \hat{\theta}^{(1)}(u) | \partial_u \hat{\psi}^{(1)}(u) \rangle \\ &+ \mathrm{i}\epsilon [E^{(2)}(u) - E^{(1)}(u)]^{-1} \langle \partial_u \hat{\theta}^{(1)}(u) | \hat{\psi}^{(2)}(u) \rangle \langle \hat{\theta}^{(2)}(u) | \partial_u \hat{\psi}^{(1)}(u) \rangle \} \\ &+ \mathcal{O}(\epsilon^2) . \end{aligned} \tag{1.5}$$

The first term corresponds to Berry's phase [5]; the second is the first-order correction in ϵ to the adiabatic approximation and thus goes beyond the results given in [1].

2. The adiabatic theorem for non-self-adjoint Hamiltonians

We assume that

$$|E^{(2)}(s) - E^{(1)}(s)| \geqslant d > 0$$

and moreover that $E^{(1)}(s)$ is the least dissipative eigenvalue, i.e.

$$\mathrm{Im}[E^{(2)}(s) - E^{(1)}(s)] \leqslant 0 .$$

It is sufficient to treat the case

$$E^{(1)}(s) \equiv 0 . \tag{2.1}$$

The general case can be reduced to this one by using the shifted Hamiltonian

$$H(s) + E^{(1)}(s)\mathbb{1} \tag{2.2}$$

which changes the Schrödinger evolution operator by the numerical factor

$$\exp\left(-\frac{\mathrm{i}}{\epsilon} \int_{s_0}^{s} E^{(1)}(u)\, \mathrm{d}u \right) . \tag{2.3}$$

Let Γ be the circle centred at the origin with radius $1/2d$, so that it does not enclose the other eigenvalue. The necessary technical smoothness condition for $H(s)$ is

$$\sup_{s,z\in\Gamma} \| \partial_s^l (H(s)-z)^{-1} \| \leqslant M_l < \infty \qquad l = 0,1,2\ldots. \tag{2.4}$$

A condition of this sort is always (explicitly or implicitly) assumed in dealing with the adiabatic expansion. For results up to and including terms of order ϵ^k it is sufficient to have (2.4) for $l \leqslant k+1$.

The projector corresponding to the eigenvalue 0 of $H(s)$ is

$$P^{(1)}(s) = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma [H(s)-z]^{-1}\,\mathrm{d}z\,.$$

We write the recurrence construction of [6, 7] adapted to the present problem in the following way. For ϵ small enough we define with $H_0(s;\epsilon) = H(s)$

$$H_{k+1}(s;\epsilon) = H_k(s;\epsilon) + B_k(s;\epsilon) \qquad (k = 0,1,2\ldots) \tag{2.5}$$

where

$$B_k(s;\epsilon) = \sum_{j=1}^{2} P_k^{(j)}(s;\epsilon)\{\mathrm{i}\epsilon \dot{P}_k^{(j)}(s;\epsilon) - [H(s), P_k^{(j)}(s;\epsilon)]\} \tag{2.6}$$

with

$$P_k^{(1)}(s;\epsilon) = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma [H_k(s;\epsilon)-z]^{-1}\,\mathrm{d}z \tag{2.7}$$

and

$$P_k^{(2)}(s;\epsilon) = \mathbf{1} - P_k^{(1)}(s;\epsilon)\,. \tag{2.8}$$

Let $U_k^A(s,s_0,\epsilon)$ be the solution of

$$\mathrm{i}\epsilon\partial_s U_k^A(s,s_0;\epsilon) = H_k^A(s;\epsilon)U_k^A(s,s_0;\epsilon) \qquad U_k^A(s_0,s_0;\epsilon) = \mathbf{1} \tag{2.9}$$

where

$$H_k^A(s;\epsilon) = H(s) - B_k(s;\epsilon) \qquad (k = 0,1,2\ldots)\,. \tag{2.10}$$

According to the theory developed in [6, 7] we have the following intertwining property

$$P_k^{(j)}(s;\epsilon)U_k^A(s,s_0;\epsilon) = U_k^A(s,s_0;\epsilon)P_k^{(j)}(s_0;\epsilon)\,. \tag{2.11}$$

We also have the estimate

$$\|B_k(s;\epsilon)\| \leqslant \epsilon^{k+1} b_k(s)\,. \tag{2.12}$$

Now let $U(s, s_0; \epsilon)$ be the solution of

$$\mathrm{i}\epsilon\partial_s U(s,s_0;\epsilon) = H(s)U(s,s_0;\epsilon) \qquad U(s_0,s_0;\epsilon) = \mathbf{1}$$

and define

$$\Omega_k(s,s_0;\epsilon) = (U_k^A)^{-1}(s,s_0;\epsilon)U(s,s_0;\epsilon)\,. \tag{2.13}$$

One easily verifies from [6, 7] that

$$\Omega_k(s,s_0;\epsilon) - \mathbf{1} = \frac{1}{\mathrm{i}\epsilon}\int_{s_0}^{s}(U_k^A)^{-1}(u,s_0;\epsilon)B_k(u;\epsilon)U_k^A(u,s_0;\epsilon)\Omega_k(u,s_0;\epsilon)\,\mathrm{d}u\,. \tag{2.14}$$

Up to now the results taken over from [6, 7] do not depend on the self-adjointness of $H(s)$ and therefore carry over to our present problem. The difference comes up if we try to use the integral equation (2.14) to estimate $\|\Omega_k - \mathbf{1}\|$. In the self-adjoint case, due to the unitarity of U_k^A, the integral equation gave at once

$$\|\Omega_k(s,s_0;\epsilon) - \mathbf{1}\| \leqslant \epsilon^k \int_{s_0}^{s} b_k(u)\,\mathrm{d}u\,.$$

If $H \neq H^\dagger$, $\|U_k^A\|$ and $\|(U_k^A)^{-1}\|$ can blow up and the estimate no longer holds.

To see which estimate is physically relevant let us first notice from the definition of B_k (2.6) that if $H(s)$ is constant outside $[s_-, s_+]$, then $P_k^{(j)}(s;\epsilon) = P^{(j)}(s_+)$, $(P^{(j)}(s_-))$ if $s \geqslant s_+$, $(s \leqslant s_-)$. Therefore, taking s_0 and s on the left and on the right of that interval, the quantity we are interested in is the probability that under the influence of $H(s)$ the eigenstate corresponding to $P^{(1)}(s_0) = P^{(1)}(s_-)$ goes over into the eigenstate of $P(s) = P(s_+)$. This amounts to the calculation of

$$P^{(1)}(s)U(s,s_0;\epsilon)P^{(1)}(s_0)\,.$$

The interval can also be taken to be the whole real axis, if $H(s)$ approaches limits sufficiently quickly; this amounts to imposing the condition

$$\int_{-\infty}^{+\infty} b_k(u)\,\mathrm{d}u < \infty\,. \tag{2.15}$$

In what follows we will consider

$$P_k^{(1)}(s;\epsilon)U(s,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon) \qquad \text{for all} \quad s \geqslant s_0\,.$$

Now, due to the definition (2.13) of Ω_k and the intertwining property (2.11)

$$\begin{aligned}
&P_k^{(1)}(s;\epsilon)U(s,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)\\
&\quad= P_k^{(1)}(s;\epsilon)U_k^A(s,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)\Omega_k(s,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)\\
&\quad= P_k^{(1)}(s;\epsilon)U_k^A(s,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)\\
&\qquad\times\left\{\mathbf{1} + P_k^{(1)}(s_0;\epsilon)[\Omega_k(s,s_0;\epsilon) - \mathbf{1}]P_k^{(1)}(s_0;\epsilon)\right\}.
\end{aligned}$$

In our framework, to establish the adiabatic approximation of order k means to prove that the second term in the curly bracket is of order ϵ^k and can be neglected. That this is possible depends on the estimate

$$\|P_k^{(1)}(s_0;\epsilon)(\Omega_k(s,s_0;\epsilon)-1)P_k^{(1)}(s_0;\epsilon)\| < c\epsilon^k \tag{2.16}$$

where c is a constant uniformly bounded in s, s_0 with $s \geqslant s_0$ and ϵ sufficiently small.

Before proving the above we give an explicit formula for $U_k^A(s,s_0;\epsilon)$. To this end we first define $\hat{\psi}_k^{(j)}(s;\epsilon)$ and $\hat{\theta}_k^{(j)}(s;\epsilon)$ by

$$\hat{\psi}_k^{(j)}(s;\epsilon) \equiv P_k^{(j)}(s;\epsilon)\hat{\psi}^{(j)}(s) \tag{2.17}$$

$$\hat{\theta}_k^{(j)}(s;\epsilon) \equiv P_k^{(j)\dagger}(s;\epsilon)\hat{\theta}^{(j)}(s)\,. \tag{2.18}$$

The $\hat{\psi}_k^{(j)}$ and $\hat{\theta}_k^{(j)}$ are eigenvectors of H_k and $H_k^\dagger$ respectively. Notice that

$$\langle\hat{\theta}_k^{(j)}(s;\epsilon)|\hat{\psi}_k^{(l)}(s;\epsilon)\rangle = 0 \qquad \text{for} \quad j \neq l\,.$$

Furthermore we define

$$\hat{\psi}_k^{(j)}(s,s_0;\epsilon) \equiv U_k^A(s,s_0;\epsilon)\hat{\psi}_k^{(j)}(s_0;\epsilon)\,. \tag{2.19}$$

From the intertwining property (2.11) and the definition (2.17) we have

$$\hat{\psi}_k^{(j)}(s,s_0;\epsilon) = \exp[-\mathrm{i}\varphi_k^{(j)}(s,s_0;\epsilon)]\hat{\psi}_k^{(j)}(s,\epsilon)\,. \tag{2.20}$$

From the differential equation (2.9) for U_k^A one gets the differential equation for $\varphi_k^{(j)}$:

$$\begin{aligned}\langle\hat{\theta}_k^{(j)}(s;\epsilon)|\hat{\psi}_k^{(j)}(s;\epsilon)\rangle\partial_s\varphi_k^{(j)}(s,s_0;\epsilon) + \mathrm{i}\langle\hat{\theta}_k^{(j)}(s;\epsilon)|\partial_s\hat{\psi}_k^{(j)}(s;\epsilon)\rangle&\\ = \frac{1}{\epsilon}\langle\hat{\theta}_k^{(j)}(s;\epsilon)|H_k^A(s;\epsilon)|\hat{\psi}_k^{(j)}(s;\epsilon)\rangle&\end{aligned} \tag{2.21}$$

$$\varphi_k^{(j)}(s_0,s_0;\epsilon) = 0\,.$$

This has the solution

$$\varphi_k^{(j)}(s,s_0;\epsilon) = \int_{s_0}^{s}\frac{\epsilon^{-1}\langle\hat{\theta}_k^{(j)}(u;\epsilon)|H_k^A(u;\epsilon)|\hat{\psi}_k^{(j)}(u;\epsilon)\rangle - \mathrm{i}\langle\hat{\theta}_k^{(j)}(u;\epsilon)|\partial_u\hat{\psi}_k^{(j)}(u;\epsilon)\rangle}{\langle\hat{\theta}_k^{(j)}(u;\epsilon)|\hat{\psi}_k^{(j)}(u;\epsilon)\rangle}\,\mathrm{d}u\,. \tag{2.22}$$

An arbitrary vector can be written as a linear combination of the $\hat{\psi}_k^{(j)}(s_0;\epsilon)$:

$$\psi = \sum_{j=1}^{2} a^{(j)}P_k^{(j)}(s_0;\epsilon)\hat{\psi}^{(j)}(s_0)\,. \tag{2.23}$$

Using (2.19) and (2.20) we see, that with (2.22) we have the explicit solution for U_k^A:

$$U_k^A(s,s_0,\epsilon)\psi = \sum_{j=1}^{2} a^{(j)} \exp\left[-\mathrm{i}\varphi_k^{(j)}(s,s_0;\epsilon)\right] P_k^{(j)}(s;\epsilon)\hat{\psi}^{(j)}(s)\,. \tag{2.24}$$

If $s_0 \leqslant s_-$ and $s \geqslant s_+$, (2.23) and (2.24) become

$$U_k^A(s,s_0,\epsilon)\sum_{j=1}^{2} a^{(j)}\hat{\psi}^{(j)}(s_-) = \sum_{j=1}^{2} a^{(j)} \exp\left[-\mathrm{i}\varphi_k^{(j)}(s,s_0;\epsilon)\right]\hat{\psi}^{(j)}(s_+)\,. \tag{2.25}$$

We now compute explicitly the first terms of $\varphi_k^{(j)}$ as an expansion in ϵ. To do this we will use the fact that due to (2.10) and (2.12)

$$H_k^A = H + \mathcal{O}(\epsilon^{k+1})\,. \tag{2.26}$$

This means that we can replace H_k^A by H in (2.21) and (2.22), if we want to calculate $\varphi_k^{(j)}$ up to and including terms of order ϵ^k. To get the expansions of $\hat{\psi}_k^{(j)}$ (and $\hat{\theta}_k^{(j)}$) we have to expand $P_k^{(j)}$ (and $P_k^{(j)\dagger}$). To this end we write

$$P_k^{(j)}(s;\epsilon) = P^{(j)}(s) + \sum_{j=1}^{\infty}\epsilon^n Q_{k,n}^{(j)}(s)$$

From (2.5), (2.7) and (2.12) we see that

$$Q_{k,n}^{(j)} = Q_{k+1,n}^{(j)} \qquad \text{for} \quad n \leqslant k\,.$$

In other words if we increase k, the low-order expansion coefficients do not change. We are in fact only interested in the expansion up to and including $n = 2$. Taking $k \geqslant 2$ the operators $Q_{k,1}$, $Q_{k,2}$ do not depend on k and we can write (also suppressing the arguments s and ϵ)

$$P_k^{(j)} = P^{(j)} + \epsilon Q_1^{(j)} + \epsilon^2 Q_2^{(j)} + \mathcal{O}(\epsilon^3) \qquad k \geqslant 2\,.$$

Using the projection property $P_k^{(j)2} = P_k^{(j)}$ one easily derives (for future use)

$$P^{(j)}Q_1^{(j)}P^{(j)} = 0 \tag{2.27}$$

$$P^{(j)}Q_2^{(j)}P^{(j)} = -P^{(j)}Q_1^{(j)2}P^{(j)}\,. \tag{2.28}$$

The last equation allows us to eliminate $Q_2^{(j)}$ in what follows. From (2.5) and (2.6) we have

$$H_1 = H + \mathrm{i}\epsilon\sum_{j=1}^{2} P^{(j)}\dot{P}^{(j)}\,.$$

From (2.7) we have

$$P_1^{(1)} = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma (H_1 - z)^{-1}\,\mathrm{d}z$$
$$= -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \left[(H-z)^{-1} - (H-z)^{-1}\mathrm{i}\epsilon \sum_{j=1}^{2} P^{(j)}\dot{P}^{(j)}(H-z)^{-1}\right]\mathrm{d}z + \mathcal{O}(\epsilon^2)$$
$$= P^{(1)} + \epsilon\, Q_1^{(1)} + \mathcal{O}(\epsilon^2)$$

with

$$Q_1^{(1)}(s) = -\frac{\mathrm{i}}{E^{(2)}(s)}[P^{(1)}(s)\dot{P}^{(1)}(s)P^{(2)}(s) + P^{(2)}(s)\dot{P}^{(2)}(s)P^{(1)}(s)]\,.$$

We have thus expressed $P_1^{(1)}(s)$ and by (2.8) also $P_1^{(2)}(s)$ up to terms of $\mathcal{O}(\epsilon^2)$ by the given quantities.

To get $\varphi_1^{(j)}$ from (2.22) up to terms of $\mathcal{O}(\epsilon^2)$ we have to calculate the denominator of the integrand up to terms of $\mathcal{O}(\epsilon^3)$. With (2.17) and (2.18) we have (again for $k \geqslant 2$)

$$\hat{\psi}_k^{(j)}(s;\epsilon) = [P^{(j)}(s) + \epsilon\, Q_1^{(j)}(s) + \epsilon^2 Q_2^{(j)}(s)]\hat{\psi}^{(j)}(s)$$
$$\hat{\theta}_k^{(j)}(s;\epsilon) = [P^{(j)\dagger}(s) + \epsilon\, Q_1^{(j)\dagger}(s) + \epsilon^2 Q_2^{(j)\dagger}(s)]\hat{\theta}^{(j)}(s)\,.$$

Using (2.27) and (2.28) one easily derives

$$\langle \hat{\theta}_k^{(j)}(s;\epsilon)|\hat{\psi}_k^{(j)}(s;\epsilon)\rangle = 1 - \epsilon^2 \langle \hat{\theta}^{(j)}(s)|Q_1^{(j)2}(s)|\hat{\psi}^{(j)}(s)\rangle + \mathcal{O}(\epsilon^3)\,. \tag{2.29}$$

Using the identity for projectors $P_k^{(j)}\dot{P}_k^{(j)}P_k^{(j)} = 0$ one can calculate

$$\langle \hat{\theta}_k^{(j)}(s;\epsilon)|\partial_s \hat{\psi}_k^{(j)}(s;\epsilon)\rangle = \langle \hat{\theta}^{(j)}(s)|\partial_s \hat{\psi}^{(j)}(s)\rangle + \epsilon[\langle \hat{\theta}^{(j)}(s)|P^{(j)}(s)\dot{Q}_1^{(j)}(s)$$
$$+ Q_1^{(j)}(s)\dot{P}^{(j)}(s)|\hat{\psi}^{(l)}(s)\rangle + \langle \hat{\theta}^{(j)}(s)|Q_1^{(j)}(s)|\partial_s \hat{\psi}^{(j)}(s)\rangle] + \mathcal{O}(\epsilon^2)\,. \tag{2.30}$$

Remembering (2.26) and using (2.27) as well as some other of the preceding relations we get

$$\langle \hat{\theta}_k^{(j)}(s;\epsilon)|H_k^A|\hat{\psi}_k^{(j)}(s;\epsilon)\rangle = E^{(j)}(s) + \epsilon^2 E_2^{(j)}(s) + \mathcal{O}(\epsilon^3)$$

with

$$E_2^{(j)}(s) = \langle \hat{\theta}^{(j)}(s)| - 2E^{(j)}(s)Q_1^{(j)2}(s) + Q_1^{(j)}(s)H(s)Q_1^{(j)}(s)|\hat{\psi}^{(j)}(s)\rangle\,. \tag{2.31}$$

Equations (2.29), (2.30) and (2.31) allow us to calculate $\varphi_k^{(1)}$ from (2.22) up to and including the linear terms in ϵ. Keeping in mind the fact that we treated the special case (2.1), we have after some algebraic manipulations for $k \geqslant 2$

$$\varphi_k^{(1)}(s,s_0;\epsilon) = \int_{s_0}^{s} \bigg(-\mathrm{i}\langle \hat{\theta}^{(1)}(u)|\partial_u \hat{\psi}^{(1)}(u)\rangle$$
$$- \frac{\epsilon}{E^{(2)}(u)}\langle \partial_u \hat{\theta}^{(1)}(u)|P^{(2)}(u)|\partial_u \hat{\psi}^{(1)}(u)\rangle + \mathcal{O}(\epsilon^2)\bigg)\,\mathrm{d}u\,. \tag{2.32}$$

Going over now to the case with $E^{(1)}(s) \neq 0$, we see that in accordance with (2.2) we have to replace $E^{(2)}(u)$ by $E^{(2)}(u) - E^{(1)}(u)$ in (2.32), where $E^{(j)}(s)$ from now on are the eigenvalues of the shifted Hamiltonian. Taking together (2.3), (2.25) and (2.32), and then comparing with (1.2) and (1.3) shows that (1.5) holds as soon as the estimate (2.16) holds, which we will prove now (for the unshifted Hamiltonian!).

A finite number of constants, uniformly bounded with respect to s and $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$, will appear during the estimation. For simplicity we shall denote all of them by the same letter c. Recall that we assume (2.4) and if $s_+ \, (s_-) = \infty \, (-\infty)$ then in addition (2.15) has to hold.

We start with (2.23), written for general s:

$$\psi = \sum_{j=1}^{2} a_k^{(j)}(s) \hat{\psi}_k^{(j)}(s;\epsilon) .$$

Since

$$a_k^{(j)}(s) = \frac{\langle \hat{\theta}_k^{(j)}(s) | \psi \rangle}{\langle \hat{\theta}_k^{(j)}(s) | \hat{\psi}_k^{(j)}(s) \rangle}$$

we have

$$|a_k^{(j)}(s)| \leqslant c \, \|\psi\| . \tag{2.33}$$

Furthermore, from (2.22), (2.29), (2.30), and (2.31) we have

$$\operatorname{Im} \varphi_k^{(j)}(s, s_0; \epsilon) \leqslant c \qquad \text{for} \quad s \geqslant s_0$$

and moreover

$$- \operatorname{Im} \varphi_k^{(1)}(s, s_0; \epsilon) \leqslant c .$$

From (2.24), (2.33) and $\|\hat{\psi}_k^{(j)}(s;\epsilon)\| < c$ it follows that for $s \geqslant u$

$$\|U_k^A(s, u; \epsilon)\| < c . \tag{2.34}$$

From (2.24) we have furthermore for $s \geqslant s_0$

$$(U_k^A)^{-1}(s, s_0; \epsilon) \sum_{j=1}^{2} a_k^{(j)}(s) \hat{\psi}_k^{(j)}(s;\epsilon) = \sum_{j=1}^{2} a_k^{(j)}(s) \exp\left[\mathrm{i} \varphi_k^{(j)}(s, s_0; \epsilon) \right] \hat{\psi}_k^{(j)}(s_0;\epsilon)$$

so that

$$P_k^{(1)}(s_0;\epsilon) U_k^{A-1}(s, s_0; \epsilon) P_k^{(1)}(s;\epsilon) |\psi\rangle = \exp\left[\mathrm{i} \varphi_k^{(1)}(s, s_0; \epsilon) \right] a_k^{(1)}(s) |\hat{\psi}_k^{(1)}(s_0;\epsilon)\rangle$$

and

$$\|P_k^{(1)}(s_0;\epsilon) U_k^{A-1}(s, s_0; \epsilon) P_k^{(1)}(s;\epsilon)\| \leqslant c . \tag{2.35}$$

Taking into account (2.14) and (2.11) in the Dyson expansion for $\Omega_k - 1$, we have

$$P_k^{(1)}(s_0;\epsilon)(\Omega_k(s,s_0;\epsilon)-1)P_k^{(1)}(s_0;\epsilon)$$
$$= \sum_{n=1}^{\infty}(\mathrm{i}\epsilon)^{-n}\int_{s_0}^{s}\mathrm{d}s_1\int_{s_0}^{s_1}\mathrm{d}s_2\cdots\int_{s_0}^{s_{n-1}}\mathrm{d}s_n$$
$$\times [P_k^{(1)}(s_0;\epsilon)U_k^{A-1}(s_1,s_0;\epsilon)P_k^{(1)}(s_1;\epsilon)$$
$$\times B_k(s_1;\epsilon)U_k^A(s_1,s_2)B_k(s_2;\epsilon)\cdots$$
$$\times \cdots U_k^A(s_{n-1},s_n;\epsilon)B_k(s_n;\epsilon)U_k^A(s_n,s_0;\epsilon)P_k^{(1)}(s_0;\epsilon)]\,.$$

This series we estimate term by term. Using (2.34), (2.35) and (2.12) we obtain

$$\|P_k^{(1)}(s_0;\epsilon)[\Omega_k(s,s_0;\epsilon)-1]P_k^{(1)}(s_0;\epsilon)\| \leqslant \sum_{n=1}^{\infty}\frac{c^{n+1}}{n!}\left(\epsilon^k\int_{s_0}^{s}b_k(u)\,\mathrm{d}u\right)^n$$

which gives the desired estimate (2.16).

3. Example and remarks

As an example we take the model (motivated by [8]) treated in [3]. The Hamiltonian in $\mathcal{H}=\mathcal{C}^2$ written in the basis

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

has the form

$$\begin{pmatrix}\omega^{(1)} & \Omega(s)\\ \Omega^*(s) & \omega^{(2)}-\frac{1}{2}\mathrm{i}\beta\end{pmatrix} \tag{3.1}$$

where $\omega^{(2)} \geqslant \omega^{(1)}$ and $\beta \geqslant 0$ are independent of s. $\Omega(s)$ is supposed to be bounded with many bounded derivatives.

The Hamiltonian given by (3.1) can be put in the form (1.1) with

$$E^{(1)}(s) = \tfrac{1}{2}\left(\omega^{(1)}+\omega^{(2)}-\tfrac{1}{2}\mathrm{i}\beta-\sqrt{\tfrac{1}{2}(r+x)}+\mathrm{i}\sqrt{\tfrac{1}{2}(r-x)}\right) \tag{3.2}$$

$$E^{(2)}(s) = \tfrac{1}{2}\left(\omega^{(1)}+\omega^{(2)}-\tfrac{1}{2}\mathrm{i}\beta+\sqrt{\tfrac{1}{2}(r+x)}-\mathrm{i}\sqrt{\tfrac{1}{2}(r-x)}\right) \tag{3.3}$$

Here

$$x=(\omega^{(2)}-\omega^{(1)})^2-\tfrac{1}{4}\beta^2+4|\Omega|^2 \qquad r=\sqrt{x^2+y^2} \qquad y=\beta(\omega^{(1)}-\omega^{(2)})\leqslant 0\,.$$

The positive square root has to be taken in these formulae and we have suppressed the dependence on s, as we will do in the following. The eigenstates become with the correct normalization:

$$|\hat{\psi}^{(1)}\rangle = N^{(1)}[\Omega^*|1\rangle+(E^{(1)}-\omega^{(1)})|2\rangle] \tag{3.4}$$

$$|\hat{\psi}^{(2)}\rangle = N^{(2)}[(E^{(2)}-\omega^{(2)}+\tfrac{1}{2}\mathrm{i}\beta)|1\rangle+\Omega|2\rangle] \tag{3.5}$$

$$|\hat{\theta}^{(1)}\rangle = N^{(3)}[\Omega^*|1\rangle+(E^{(1)*}-\omega^{(1)})|2\rangle] \tag{3.6}$$

$$|\hat{\theta}^{(2)}\rangle = N^{(4)}[(E^{(2)*}-\omega^{(2)}-\tfrac{1}{2}\mathrm{i}\beta)|1\rangle+\Omega|2\rangle] \tag{3.7}$$

with

$$N^{(1)}N^{(3)*} = [|\Omega|^2 + (E^{(1)} - \omega^{(1)})^2]^{-1} \tag{3.8}$$

$$N^{(2)}N^{(4)*} = [|\Omega|^2 + (E^{(2)} - \omega^{(2)} + \tfrac{1}{2}\mathrm{i}\beta)^2]^{-1} \,. \tag{3.9}$$

The main conclusion is that from (3.2) and (3.3) it follows that

$$\mathrm{Im}[E^{(2)}(s) - E^{(1)}(s)] \leqslant 0 \qquad \text{for all } s$$

so that the theory developed in the previous section applies. Notice that for $\beta \neq 0$ the equality is reached only if $\omega^{(1)} = \omega^{(2)}$ and $|\Omega| \geqslant \frac{1}{4}\beta$.

Assuming that $\Omega(s) = 0$ for $s \leqslant s_-$ and $s \geqslant s_+$ and inserting (3.2) to (3.9) into (1.5) we have $c^{(1)}(s)$ explicitly. Multiplying by the factor (2.3) we have the probability

$$\left| c^{(1)}(s) \exp\left(-\frac{\mathrm{i}}{\epsilon} \int_{s_-}^{s_+} E^{(1)}(u)\,\mathrm{d}u \right) \right|^2$$

that if the system is in the state $|1\rangle$ at $s = s_-$ it will be in that state for $s \geqslant s_+$.

As discussed already in the introduction we did not make use of the scaling assumption used in [3], namely that β is of order ϵ.

Notice also that no condition is imposed on $\mathrm{Im}\, E^{(1)}$, so that the results apply for 'pumping' systems as well.

We end with a remark of a general nature. Assume, for the moment, that in (1.1) the $E^{(j)}(s)$ are real and $\hat{\psi}^{(j)}(s) = \hat{\theta}^{(j)}(s)$, so that $H(s) = H^\dagger(s)$. Assume moreover that $H(s)$ is analytic in a strip around the real axis and that it approaches limits for $s \to \pm\infty$ *sufficiently fast*. Then, as has been proved in [9] (for related results of this sort see [10–12]) the adiabatic transitions are exponentially suppressed. This means if at all $t = -\infty$ the system was in the state corresponding to $E^{(1)}(-\infty)$, then at $t = +\infty$ the system will be in the state corresponding to $E^{(1)}(+\infty)$ with probability $1 + \mathcal{O}(\exp(-c/\epsilon)), c > 0$. Suppose now that we replace $E^{(2)}(s)$ by $E^{(2)}(s) - \mathrm{i}\beta$, $\beta > 0$, so that the corresponding level is unstable. Naively one might believe that since the two levels are adiabatically decoupled the result about level 1 remains valid. Actually this is not so since our results show that in this case starting again with the state corresponding to $E^{(1)}(-\infty)$ the probability of finding the system in the state corresponding to $E^{(1)}(+\infty)$ is $1 - a\epsilon + \mathcal{O}(\epsilon^2)$, $a > 0$.

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